Differential Geometry I: Curves and surfaces

A. Standard exercises:

9.1 (Regular curve as an intersection of transversal surfaces) Let $C \subset \Omega$ be the set of points in Ω such that

$$f(x,y,z) = g(x,y,z) = 0,$$

where Ω is a domain of $\mathbb{R}^3$ and $f, g \in C^k(\Omega)$ with $k \geqslant 1$. Assume that for a point $(x_0, y_0, z_0) \in C$, the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

has rank 2.

(a) Show that the subsets $\{f = 0\} \subset \mathbb{R}^3$ and $\{g = 0\} \subset \mathbb{R}^3$ define, in a neighborhood of (x_0, y_0, z_0), C^k surfaces that intersect transversally (see Ex. 8.4).

(b) Show that one can locally parametrize the set C in a neighborhood of (x_0, y_0, z_0) as a regular curve $\gamma : I \to \Omega$ of class C^k.

(b) Find an expression for the tangent vector $\dot{\gamma}(t)$.

Solution. **(a)** Since the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

has rank 2 at the point (x_0, y_0, z_0), each of its rows is a non zero vector. In particular, $\nabla f(x_0, y_0, z_0) \neq 0$ and $\nabla g(x_0, y_0, z_0) \neq 0$. Since ∇f and ∇g are continuous, there exists a small neighorhood $\mathcal{W}$ of (x_0, y_0, z_0) such that $\nabla f \neq 0$ and $\nabla g \neq 0$ on $\mathcal{W}$. By the regular value theorem, this means that $S_f = \{f = 0\} \cap \mathcal{W}$ and $S_g \{g = 0\} \cap \mathcal{W}$ are smooth codimension 1 submanifolds of $\mathbb{R}^3$, i.e. surfaces.

Let $F : \Omega \to \mathbb{R}^2$ be defined by $F = (f, g)$. By hypothesis, F is of class C^k and has maximal rank 2 at (x_0, y_0, z_0). By lower semicontinuity of rank, F has constant rank 2 in a neighborhood $\mathcal{U}$ of that point (which, by restricting to a smaller neighborhood if necessary, we can assume without loss of generality that it is contained in $\mathcal{W}$). Thus, at every point p in $\mathcal{U}$, the vectors $\nabla f(p)$ and $\nabla g(p)$ are linearly independent; therefore, the planes $\nabla f(p)^\perp$ and $\nabla g(p)^\perp$ are not the same plane; in particular, $\nabla f(p)^\perp + \nabla g(p)^\perp = \mathbb{R}^3$ (since $\nabla f(p)^\perp$ contiains a vector v not in $\nabla g p)^\perp$, so v and $\nabla g(p)^\perp$ span all of $\mathbb{R}^3$). In particular, if $p \in S_f \cap S_g$, then $T_p S_f + T_p S_g = \nabla f(p)^\perp + \nabla g(p)^\perp = \mathbb{R}^3$, so S_f and S_g intersect transversally.

(b) We give two proofs of the existence of a parametrization of C as a curve of class C^k.

First proof. Let $F = (f, g)$ be the map defined in part (a) and $\mathcal{U}$ a neighborhood of (x_0, y_0, z_0) where F has rank 2. Applying the regular value theorem, we conclude that $F^{-1}(0) \cap \mathcal{U} = C \cap \mathcal{U}$ is a C^k submanifold near (x_0, y_0, z_0). Its codimension is 2, so its dimension is $1 = 3 - 2$. Therefore, there exists a (possibly smaller) neighborhood $V \subset \mathbb{R}^3$ of (x_0, y_0, z_0) and a bijective regular parametrization $\gamma : I \to V \cap C$ of class C^k.

Second proof. Since F has constant rank 2 around (x_0, y_0, z_0), by applying the constant rank theorem, there exist:

(i) A neighborhood $U \subset \Omega$ of (x_0, y_0, z_0) and a diffeomorphism $\Phi : U \to \mathbb{R}^3$ with $\Phi(x_0, y_0, z_0) = (0, 0, 0)$.

(ii) A neighborhood $V \subset \mathbb{R}^2$ of $F(x_0, y_0, z_0)$ and a diffeomorphism $\Psi : V \to \mathbb{R}^2$ with $\Psi(F(x_0, y_0, z_0)) = (0, 0)$.

(iii) A linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ of rank 2 such that $L \circ \Phi = \Psi \circ F$.

Thus $(x, y, z) \in C \cap U$ iff $L(\Phi(x, y, z)) = 0$, i.e. $\Phi(x, y, z) \in \ker L$.
The kernel of L is 1-dimensional, generated by a nonzero vector $v \in \mathbb{R}^3$. Then

$$\gamma(t) = \Phi^{-1}(tv)$$

is a local parametrization of C near (x_0, y_0, z_0).

(c) Since γ lies on the surface $f = 0$, its velocity vector $\dot{\gamma}$ is orthogonal to ∇f. Similarly it is orthogonal to ∇g:

$$\langle \nabla f(\gamma(t)), \dot{\gamma}(t)\rangle = \langle \nabla g(\gamma(t)), \dot{\gamma}(t)\rangle = 0.$$

Hence $\dot{\gamma}(t)$ is collinear with the cross product

$$\nabla f(\gamma(t)) \times \nabla g(\gamma(t)).$$

9.2 Recall what it means for a map $f : \mathcal{M} \to \mathcal{N}$ between two smooth submanifolds to be smooth and what is the differential $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$. Prove the composition rule for derivatives: If $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{L}$ are smooth maps between submanifolds, then for any $p \in \mathcal{M}$:

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Solution. If $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{L}$ are smooth maps between the submanifolds $\mathcal{M} \subseteq \mathbb{R}^m$, $\mathcal{N} \subseteq \mathbb{R}^n$ and $\mathcal{L} \subseteq \mathbb{R}^l$, then, for any $p \in \mathcal{M}$, there exists a neighborhood $\mathcal{U}$ of p in $\mathbb{R}^m$ and $\mathcal{V}$ of $f(p)$ in $\mathbb{R}^n$, as well as smooth extensions $F : \mathcal{U} \to \mathbb{R}^n$ (of f) and $g : \mathcal{V} \to \mathbb{R}^l$ (of g). In this case, by the differentiation of the differentials, $df_p = dF_p|_{T_p\mathcal{M}}$ and $dg_{f(p)} = dG_{f(p)}|_{T_{f(p)}\mathcal{N}}$. By using the usual composition rule for derivatives for functions defined on open domains of $\mathbb{R}^k$, we have:

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p$$

and, therefore, since $F(p) = f(p)$, $dF_p|_{T_p(\mathcal{M})} = df_p$ and $dG_{f(p)}|_{Im(df_p} = dg_{f(p)}$ (since $Im(df_p) \subset T_{f(p)}\mathcal{N}$) we infer that

$$d(G \circ F)_p|_{T_p\mathcal{M}} = dG_{F(p)} \circ dF_p|_{T_p\mathcal{M}} = dG_{f(p)} \circ df_p = dg_{f(p)} \circ df_p.$$

9.3 Recall that by definition a map $f : \mathcal{M} \to \mathcal{N}$ between two smooth submanifolds is a smooth *diffeomorphism* if it is bijective and both f and f^{-1} are smooth (for the purposes of this exercise, smooth means at least C^1).

(a) Prove that for every $p \in \mathcal{M}$, the differential $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ is an isomorphism of vector spaces.

(b) Deduce that there cannot exist a smooth diffeomorphism between two non-empty manifolds that do not have the same dimension.

(c) Show, by an example, that a bijective smooth map $f : \mathcal{M} \to \mathcal{N}$ between two smooth submanifolds is not necessarily a diffeomorphism (you may assume $\dim(\mathcal{M}) = 1$).

(*d) Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map. Let $p \in \mathcal{M}$ be such that $df_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ is a bijective map. Show that f is locally a diffeomorphism around p.

Solution. **(a)** Let $q = f(p)$ and $g = f^{-1} : N \to M$. Then $g \circ f : M \to M$ is the identity, so its differential

$$d(g \circ f)_p : T_pM \to T_pM$$

is also the identity. The chain rule gives

$$dg_q \circ df_p = d(g \circ f)_p = \mathrm{Id}_{T_pM}.$$

Similarly $df_p \circ dg_q = \mathrm{Id}_{T_qN}$. Thus df_p is an isomorphism.

(b) If $f : M \to N$ is a diffeomorphism, then $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for all p, so

$$\dim(M) = \dim(T_pM) = \dim(T_{f(p)}N) = \dim(N).$$

(c) It suffices to find a differentiable bijection for which df_p fails to be an isomorphism at some point. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a C^∞ bijection but $f'(0) = 0$, so it is not a diffeomorphism.

Remark. There does not exist a homeomorphism between manifolds of different dimensions either (Brouwer's invariance of domain theorem). However, Cantor showed that for any $m, n \geqslant 1$ there exists a bijection $\mathbb{R}^m \to \mathbb{R}^n$.

9.4 If you haven't already done so previously (since the corresponding exercises in sheet 7 and 8 were marked with an asterisk), prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$ and describe the tangent space $T_IO(n)$ of this manifold at the point I (the identity matrix). Show that the exponential matrix map $A \to \exp(A)$ maps $T_IO(n)$ to $SO(n)$ (use Ex. 3.7); use this map to construct a local parametrization of $O(n)$ around I (you might want to use part (d) of the previous exercise).

Solution. As we saw in the previous exercise sheets, Near $O(n)$, the map

$$\Phi : A \mapsto AA^T$$

has constant rank. Its differential at $A = I$ is

$$d\Phi_I(H) = H + H^T.$$

Hence

$$T_I O(n) = \ker d\Phi_I = \{H \in M_n(\mathbb{R}) \mid H + H^T = 0\},$$

the space of antisymmetric matrices (which we will denote by $\mathcal{A}_n(\mathbb{R})$).

Let us consider the map $h : \mathcal{A}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$, $h(A) = \exp(A)$. As we have seen in exercise 3.7, for any antisymmetric matrix H, the matrix $\exp(H)$ is in $SO(n)$, so h maps $\mathcal{A}_n(\mathbb{R})$ to $SO(n)$. Its differential at 0 can be computed as follows: For any antisymmetric matrix H with $\|H\| < 1$:

$$h(0 + H) - h(0) = \exp(H) - I = \sum_{n=0}^{\infty} H^n - I = H + O(\|H\|^2)$$

so

$$dh_0(H) = H,$$

i.e. $dh_0 : T_0\mathcal{A}_n(\mathbb{R}) = \mathcal{A}_n(\mathbb{R}) \to T_I SO(n) = \mathcal{A}_n(\mathbb{R})$ is a linear isomorphism (its the identity map). Thus, by the inverse function theorem, there exists a neighborhood $\mathcal{U}$ of 0 in $\mathcal{A}_n(\mathbb{R})$ which is mapped diffeomorphically to $h(\mathcal{U}) \subset SO(n)$.

9.5 A surface is said to be *ruled* if it is a union of straight lines. More precisely, let $\gamma : I \to \mathbb{R}^3$ be a C^1 curve and $b : I \to \mathbb{R}^3$ a C^1 vector field along γ, then the associated ruled surface is defined by the parametrization:

$$\psi(u, v) = \gamma(u) + vb(u).$$

(a) Give the necessary and sufficient conditions for such a ruled surface to be locally a regular surface (that is, for the map ψ to be an immersion).

(b) Let C be a curve in $\mathbb{R}^3$. A *cone* with vertex $q \in \mathbb{R}^3$ and base C is the union of lines passing through q and a point of C. Give the necessary and sufficient conditions for a cone to be a regular surface in a neighborhood of its base. Then give an explicit parametrization of this cone.

(c) Explain what a Möbius strip is and provide a parametrization of this surface as a ruled surface in $\mathbb{R}^3$.

Solution. **(a)** The map ψ is C^1. It is an immersion iff

$$\frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} = \dot{\gamma}(u) \times b(u) + v\,\dot{b}(u) \times b(u) \neq 0.$$

In particular $\dot{\gamma}(u) \times b(u) \neq 0$, i.e. $\dot{\gamma}(u)$ and $b(u)$ are linearly independent. For small v, this condition is sufficient.

(b) A cone is the special case $b(u) = q - \gamma(u)$. Regularity requires that the tangent vector $\dot{\gamma}(u)$ never points toward q, i.e. $\dot{\gamma}(u)$ is not a multiple of $q - \gamma(u)$. A parametrization is

$$\psi(u,v) = \gamma(u) + v(q - \gamma(u)) = vq + (1-v)\gamma(u).$$

(c) A Möbius strip is obtained by taking a rectangle

$$Q = \{(u,v) \in \mathbb{R}^2 \mid 0 \leqslant u \leqslant \ell,\ -\varepsilon < v < \varepsilon\},$$

and identifying $(0,v) \sim (\ell,-v)$. One can construct such a strip physically. It can be realized as a ruled surface.

Let $\ell = 2\pi$ and

$$\gamma(u) = (\cos u,\, \sin u,\, 0).$$

Choose a unit vector field b orthogonal to $\dot{\gamma}$ and rotating half a turn over one revolution:

$$b(u) = \cos\left(\frac{u}{2}\right)(\cos u, \sin u, 0) + \sin\left(\frac{u}{2}\right)(0,0,1).$$

Thus the M̈bius strip is parametrized by

$$\psi(u,v) = \gamma(u) + v\,b(u), \qquad (u,v) \in [0,2\pi] \times [-\varepsilon,\varepsilon],$$

i.e.

$$\begin{cases} x(u,v) = \cos u + v\cos(u/2)\cos u, \\ y(u,v) = \sin u + v\cos(u/2)\sin u, \\ z(u,v) = v\sin(u/2). \end{cases}$$

9.6 Show that the one-sheeted hyperboloid H given by $x^2 + y^2 - z^2 = 1$ is a doubly ruled surface (i.e., ruled in two different ways), and then give a regular parametrization of this surface based on one of these rulings.

Hint: Write the equation as $x^2 - 1 = z^2 - y^2$ and factorize. Deduce algebraically the equation of a line contained in H, then parametrize it and rotate it around the Oz-axis.

Solution. Factorize

$$x^2 + y^2 - z^2 = 1 \quad \Longrightarrow \quad (x+1)(x-1) = -(y-z)(y+z).$$

The line $x = 1$, $y = z$ satisfies the equation (parametrize it by $(x,y,z) = (1,v,v)$). Rotating this line by angle u about the z-axis gives

$$\begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ v \\ v \end{pmatrix} = (\cos u - v\sin u,\ \sin u + v\cos u,\ v),$$

which again satisfies the hyperboloid equation. Thus

$$\psi(u,v) = (\cos u - v\sin u,\ \sin u + v\cos u,\ v)$$

is a parametrization of H.

The alternative ruling (with $y = -z$) gives

$$\psi(u,v) = (\cos u - v\sin u,\ \sin u + v\cos u,\ -v).$$

B. Bonus exercise:

9.7 In this exercise, we construct an example of an injective immersion that is not an embedding. The *lemniscate of Gerono* is the plane curve defined by the equation

$$4x^2 - 4y^2 - x^4 = 0,$$

that is, the set

$$C = \{(x,y) \in \mathbb{R}^2 \mid 4x^2 - 4y^2 - x^4 = 0\}.$$

(a) Show that C is not a differentiable submanifold of $\mathbb{R}^2$.

(b) Is the restriction of this curve to $\mathbb{R}^2 \setminus \{(0,0)\}$ a differentiable submanifold?

(c) Verify that $\gamma : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}^2$ defined by

$$\gamma(t) = (2\cos t, \sin(2t))$$

is a regular parametrization of C. More precisely, prove that:

(i) γ is an immersion of the open interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ into the plane.

(ii) γ is injective.

(iii) γ defines a bijection between the open interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and the curve C.

(iv) Explain what happens to γ as $t \to -\frac{\pi}{2}$ and $t \to \frac{3\pi}{2}$.

(v) Prove that γ is not an embedding of the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ into the plane (that is, it is not a homeomorphism onto its image).

Solution.

Let $f(x,y) = 4x^2 - 4y^2 - x^4$. Then $\nabla f(x,y) = (8x - 4x^3,\ -8y)$, so $(0,0)$ is the only critical point and $0 = f(0,0)$ is the only critical value.

(a) Since 0 is a critical value, the submersion theorem cannot be applied. Consider small neighborhoods

$$C_\varepsilon = \{(x,y) \in C \mid |x| < \varepsilon,\ |y| < \varepsilon\}.$$

One checks that

$$C_\varepsilon \setminus \{(0,0)\} = C_\varepsilon^{++} \cup C_\varepsilon^{+-} \cup C_\varepsilon^{-+} \cup C_\varepsilon^{--},$$

four disjoint arcs. Removing one point from a 1-dimensional manifold gives at most two connected components, so C is not a manifold.

A second argument: Consider the curves

$$\alpha_1(t) = \left(t,\ t\sqrt{1 - t^4/4}\right), \qquad \alpha_2(t) = \left(t,\ -t\sqrt{1 - t^4/4}\right).$$

Both lie in C for $|t| < 2$ and

$$\dot{\alpha}_1(0) = (1, 1), \qquad \dot{\alpha}_2(0) = (1, -1),$$

two linearly independent tangent vectors at $(0, 0)$, contradicting $\dim T_{(0,0)}C = 1$.

(b) On $\mathbb{R}^2 \setminus \{(0, 0)\}$ the function f is a submersion, so $C \setminus \{(0, 0)\}$ is a 1-dimensional submanifold.

(c) **(i)** $\dot{\gamma}(t) = (-2 \sin t,\ 2 \cos(2t))$ is never zero on the given interval.

(ii) If $\gamma(t) = \gamma(u)$, then $\cos t = \cos u$ and $\sin(2t) = \sin(2u)$, which forces $t = u$ in the interval.

(iii) One checks $f(\gamma(t)) = 0$, so γ maps into C. For any $(x, y) \in C$ with $|x| \leqslant 2$ one may choose t with $x = 2 \cos t$ and $y = \sin(2t)$ of the correct sign, giving surjectivity.

(iv) Both limits satisfy

$$\lim_{t \to -\pi/2} \gamma(t) = \lim_{t \to 3\pi/2} \gamma(t) = (0, 0).$$

The endpoint is a triple point in the continuous extension.

(v) Let $t_k = 3\pi/2 - 1/k$. Then t_k does not converge, but $\gamma(t_k)$ does. Hence γ^{-1} is not continuous and γ is not an embedding.